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# Characterization of Intertemporal Optimality in Terms of Decentralizable Conditions: The Discounted Case

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The paper studies the problem of characterizing the optimality of competitive programs in terms of "decentralizable" conditions. We show that, when future utilities are discounted, and the optimal stationary stock is proportionately expansible, then optimality of competitive programs can be characterized by the condition that the scalar product of the difference of prices and quantities, between those of the given competitive program and those of the optimal stationary program, be non-positive period by period. *Jornal of Economic Literature* Classification Number: 111. 

## 1. INTRODUCTION

The aim of this paper is to present some results on the characterization of optimality of "competitive" programs, in terms of a "decentralizable" condition, in the context of a standard Ramsey-type multisector growth model, where the technology, the period welfare function, and the period discount factor, which is assumed *less than one*, are stationary over time.

It is well known [from the price characterization results of Cass and Majumdar [2], Peleg [8, 9], Peleg and Ryder [10], Peleg and Zilcha [12], and Weitzman [13]] that in (discounted and undiscounted)

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multisectoral growth models, a program, from positive initial stocks, is optimal if and only if there exists an associated sequence of prices such that (a) these prices "support" the welfare function and the technology, at the consumption and the input–output vectors respectively, in each period, along the given program; and (b) the value of inputs at these prices converges to zero along the program (this being the relevant condition in the "discounted" case, that is, when the discount factor is less than one; in the "undiscounted" case, that is, when the discount factor equals one, the relevant condition is that the input values be uniformly bounded along the program). Condition (a) above is typically interpreted as maximization of profits and maximization of welfare (subject to an appropriate budget constraint) at the input–output vector and the consumption vector, respectively, period by period, along the given program (see Gale and Sutherland [5]).

Recently Brock and Majumdar [1] have investigated the possibility of characterizing the optimality of competitive programs in terms of conditions which can be verified by agents in an "informationally decentralized" mechanism. For a detailed discussion of the problem, we refer the reader to their paper. For our purpose here, it is sufficient to note that their objective is to replace the transversality condition, [condition, (b) above], in characterizing optimality of competitive programs, by a condition which can be verified by the agents, period by period, on the basis of information regarding prices in each period and possibly some additional fixed information or finite "messages" which are transmitted in each period.

Brock and Majumdar [1] show that in models, where there exists an "optimal stationary program" (abbreviated as o.s.p.), which has a "stationary price support," a characterization of optimality of competitive programs along such lines is possible. More specifically, they show that *in the undiscounted case*, if the input stock along the o.s.p. is "expansible" (that is, an output vector can be produced from such a stock, which provides more of each good than was provided in the initial stock), then the transversality condition (b) can be replaced by the condition that (c) the scalar product of the difference of prices and of quantities, between those of the given competitive program and those of the o.s.p., be nonpositive period by period.

The main result of our paper is to show that, *in the discounted case*, optimality of competitive programs can be characterized by (the natural analog) of the condition proposed by Brock and Majumdar. More precisely, we show that [when the discount factor is less than one], and the input stock along the o.s.p. is "proportionately expansible" [which is a slightly weaker requirement than being "expansible"], then the transversality condition (b) can be replaced by the "decentralizable condition" (c) in characterizing optimality of competitive programs.

It should be noted at this point that in the usual characterization results, it is well known that the transversality condition can be replaced by the condition that (b′) the prices "support" the value functon in each period (see Proposition 2.3 below). It is also known that (b′) implies (c) (see Theorem 2.2 below), so that the novel aspect of the new characterization results is that condition (c) is *sufficient* to guarantee that a competitive program is optimal (Theorem 3.1).

It may be of interest to observe a difference in the nature of the characterization results involving the transversality condition and those involving condition (c). Using the transversality condition, optimality or non-optimality can be verified, loosely speaking, only "at infinity," that is, by investigating the *asymptotic* behaviour of input value along a competitive program. In contrast, using condition (c), non-optimality (but not optimality) can always be detected within *some finite* horizon. A "price" is paid for this "gain." In the characterization involving the transversality condition, only knowledge regarding the competitive program is required, that is, the quantities as well as the supporting prices along the program. In contrast, in the characterization results in this paper involving condition (c), knowledge is required, in addition, of the quantities and prices of the o.s.p. Furthermore, it seems that typically, some mild "regularity" assumptions need to be made regarding the o.s.p.

Finally, it may be worthwhile to note that, like the characterization results involving condition (c), the standard characterizations using (b′) also show that non-optimality can always be detected within some finite horizon. Here, however, the difference in the two types of results lies in the extent and nature of the *information* required in the conditions (b′) and (c), respectively. In (b′), knowledge is required effectively of the entire value function, whereas in (c), the agents need to know only a finite set of numbers, namely the constant output and the constant "current prices" of the o.s.p. Furthermore, the knowledge of the value function is just a step away from a parametric solution of the intertemporal optimization problem, namely the optimal policy function. In contrast, (c) requires the knowledge of the optimal policy function at a single point, namely the initial stock of the o.s.p.

## 2. Preliminaries

### 2a. *Notation*

$R^n$ denotes the $n$-dimensional real vector space of $n$-tuples of real numbers. For $x$, $y$ in $R^n$, $x \geqslant y$ means $x_i \geqslant y_i$ for $i = 1, ..., n$; $x > y$ means $x \geqslant y$ and $x \neq y$; $x \gg y$ means $x_i > y_i$ for $i = 1, ..., n$. $R^n_+$ denotes the set $\{x$ in

$R^n: x \geqslant 0\}$, and $R^n_{++}$ denotes the set $\{x \text{ in } R^n: x \gg 0\}$. For $x$ in $R^n$ the sum norm of $x$ (denoted by $\|x\|$) is defined by $\|x\| = (\sum_{i=1}^n |x_i|)$. We denote the vector $(1, 1, ..., 1)$ in $R^n$ by $e$.

2b. *The Model*

The model is described by a triplet $(\Omega, w, \delta)$, where $\Omega$, a subset of $R^n_+ \times R^n_+$, is the *technology set*, $w: R^n_+ \to R$ is the period *walfare function*, and $\delta$ is the *discount factor* satisfying $0 < \delta < 1$. Points in $\Omega$ are written as an ordered pair $(x, y)$, where $x$ stands for the (initial) stock of inputs and $y$ stands for the (final) output which can be produced with inputs $x$.

We shall need the following assumptions on $\Omega$ and $w$.

(A.1) (a) $(0, 0)$ *is in* $\Omega$; (b) $(0, y)$ *is in* $\Omega$ *implies* $y = 0$.

(A.2) $\Omega$ *is closed.*

(A.3) *There exists a number* $\beta_0 > 0$ *such that, if* $(x, y)$ *is in* $\Omega$ *and* $\|x\| \geqslant \beta_0$, *then* $\|y\| \leqslant \|x\|$.

(A.4) *If* $(x, y)$ *is in* $\Omega$, $x' \geqslant x$, *and* $0 \leqslant y' \leqslant y$ *then* $(x', y')$ *is in* $\Omega$.

(A.5) $\Omega$ *is convex.*

(A.6) $w$ *is continuous.*

(A.7) *If* $c, c'$ *are in* $R^n_+$ *and* $c \geqslant c'$, *then* $w(c) \geqslant w(c')$; $w(e) > w(0)$.

(A.8) $w$ *is concave.*

2c. *Programs*

A *program* from $\tilde{y}$ in $R^n_+$ is a sequence $\langle x(t), y(t) \rangle$ such that

$$y(0) = \tilde{y}; \qquad 0 \leqslant x(t) \leqslant y(t) \qquad \text{and} \qquad (x(t), y(t+1)) \text{ is in } \Omega \qquad \text{for} \quad t \geqslant 0.$$

Associated with a program $\langle x(t), y(t) \rangle$ from $\tilde{y}$ is a *consumption sequence* $\langle c(t) \rangle$ defined by

$$c(t) = y(t) - x(t) \qquad \text{for} \quad t \geqslant 0.$$

To proceed further, we need the familiar preliminary result that programs from $\tilde{y}$ are uniformly bounded by a number which depends only on $\tilde{y}$ and $\beta_0$. We first establish

LEMMA 2.1. *Under* (A.3) *and* (A.4), *if* $(x, y)$ *is in* $\Omega$, *then*

(i) $\|x\| \leqslant \beta_0$ *implies* $\|y\| \leqslant \beta_0$; (ii) $\|y\| \leqslant \text{Max}\{\|x\|, \beta_0\}$.

*Proof.* (i) Suppose on the contrary there exists $(x^0, y^0)$ in $\Omega$ satisfying $\|x^0\| \leqslant \beta_0$ and $\|y^0\| > \beta_0$. Define $x' \equiv x^0 + [\beta_0 - \|x^0\|](e/n)$. Since $\beta_0 \geqslant \|x^0\|$, therefore, $x' \geqslant x^0$. Hence, by (A.4), $(x', y^0)$ is in $\Omega$. But

$\|x'\| = \beta_0$ and hence, $\|y^0\| > \|x'\|$. This contradicts (A.3) and, therefore, establishes (i).

(ii) If $\|x\| \leqslant \beta_0$, then by (i), $\|y\| \leqslant \beta_0 \leqslant \text{Max}\{\|x\|, \beta_0\}$. If $\|x\| \geqslant \beta_0$ then, by (A.3), $\|y\| \leqslant \|x\| \leqslant \text{Max}\{\|x\|, \beta_0\}$. This establishes (ii). ▮

LEMMA 2.2. *Let $\tilde{y}$ in $R^n_+$ be given. Define $B \equiv \text{Max}\{\|\tilde{y}\|, \beta_0\}$. Under* (A.3) *and* (A.4), *if $\langle x(t), y(t)\rangle$ is a program from $\tilde{y}$ then* $(\|x(t)\|, \|y(t)\|, \|c(t)\|) \leqslant (B, B, B)$, *for $t \geqslant 0$.*

*Proof.* Since $0 \leqslant c(t) = y(t) - x(t) \leqslant y(t)$ for $t \geqslant 0$, therefore, we only need to show that

$$(\|x(t)\|, \|y(t)\|) \leqslant (B, B) \qquad \text{for} \quad t \geqslant 0. \tag{2.1}$$

First, $\|x(0)\| \leqslant \|y(0)\| = \|\tilde{y}\| \leqslant B$. Hence, (2.1) holds for $t = 0$. Consider any integer $\tau > 0$. Suppose (2.1) holds for $t = \tau$. Then $\|x(\tau+1)\| \leqslant \|y(\tau+1)\| \leqslant \text{Max}\{\|x(\tau)\|, \beta_0\}$ (using Lemma 2.1(ii)) $\leqslant B$ (since $\|x(\tau)\| \leqslant B$ by hypothesis). Thus, (2.1) holds for $t = \tau + 1$. This establishes (2.1) by induction and, therefore, the lemma. ▮

2d. *Optimal and Competitive Programs*

In view of Lemma 2.2 and continuity of $w$ it is clear that for every program $\langle x(t), y(t)\rangle$ from $\tilde{y}$, $\sum_{t=0}^{\infty} \delta^t w(c(t))$ is absolutely convergent. We may, therefore, make the following definition: a program $\langle x^0(t), y^0(t)\rangle$ from $\tilde{y}$ is an *optimal program* if, for every program $\langle x(t), y(t)\rangle$ from $\tilde{y}$, we have

$$\sum_{t=0}^{\infty} \delta^t w(c^0(t)) \geqslant \sum_{t=0}^{\infty} \delta^t w(c(t)).$$

The following lemma is a standard consequence of Lemma 2.2 and is stated without proof.

LEMMA 2.3. *Under* (A.1)–(A.4), *and* (A.6), *if $\tilde{y}$ is in $R^n_+$ then there is an optimal program $\langle x(t), y(t)\rangle$ from $\tilde{y}$.*

We may then define the *value function* $V: R^n_+ \to R$ by $V(y) = \sum_{t=0}^{\infty} \delta^t w(c(t))$, where $\langle c(t)\rangle$ is the consumption sequence associated with some optimal program $\langle x(t), y(t)\rangle$ from $y$.

A sequence $\langle x(t), y(t), p(t)\rangle$ from $\tilde{y}$ is a *competitive program* if $\langle x(t), y(t)\rangle$ is a program from $\tilde{y}$, $p(t)$ is in $R^n_+$ for $t \geqslant 0$, and

$$\delta^t w(c(t)) - p(t)\, c(t) \geqslant \delta^t w(c) - p(t)\, c \qquad \text{for} \quad c \text{ in } R^n_+, \quad t \geqslant 0 \tag{2.2}$$

and

$$p(t+1)\,y(t+1)-p(t)\,x(t)\geqslant p(t+1)\,y-p(t)\,x \qquad \text{for} \quad (x,y) \text{ in } \Omega, \quad t\geqslant 0. \tag{2.3}$$

Adding up the inequalities (2.2) and (2.3), we note that if $\langle x(t), y(t), p(t)\rangle$ is a competitive program, then

$$\delta^t w(c(t))+p(t+1)\,y(t+1)-p(t)\,y(t)\geqslant \delta^t w(c)+p(t+1)\,y-p(t)(x+c) \tag{2.4}$$

for all $(x, y)$ in $\Omega$ and all $c$ in $R^n_+$.

A competitive program $\langle x(t), y(t), p(t)\rangle$ is said to satisfy the *transversality condition* if

$$\lim_{t\to\infty} p(t)\,x(t)=0. \tag{2.5}$$

2e. *Characteriation of Optimality of Competitive Programs in Terms of a Transversality Condition*

It is well known that a competitive program satisfying the transversality condition is optimal. We state this in Proposition 2.1 below for ready reference. The proof of this proposition is standard and therefore omitted. It need only be noted that (A.3), (A.4), and (A.6) guarantee (by virtue of Lemma 2.2) absolute convergence of $\sum_{t=0}^{\infty} \delta^t w(c(t))$ for any program $\langle x(t), y(t)\rangle$ from $\tilde{y}$ in $R^n_+$.

PROPOSITION 2.1. *Under* (A.3), (A.4), *and* (A.6), *if* $\langle x(t), y(t), p(t)\rangle$ *is a competitive program from* $\tilde{y}$ *in* $R^n_+$ *and*

$$\liminf_{t\to\infty} p(t)\,x(t)=0 \tag{2.6}$$

*then* $\langle x(t), y(t)\rangle$ *is an optimal program from* $\tilde{y}$.

The converse of the above proposition (Proposition 2.2 below) requires the use of the convex structure of the model. A version of it has been established [under somewhat different assumptions than the ones we use] by Peleg [8] and Peleg and Ryder [10]. The version we report here can be obtained from the result of Weitzman [13], and is proved in Dasgupta and Mitra [4].

A vector $x$ in $R^n_+$ is *sufficient* if there is $y$ in $R^n_{++}$ such that $(x, y)$ is in $\Omega$. We shall need

(A.9) *There exists a sufficient vector in* $R^n_+$.

PROPOSITION 2.2. *Under* (A.1)–(A.9) *if* $\langle x(t), y(t)\rangle$ *is an optimal* *program from* $\tilde{y}$ *in* $R^n_{++}$, *then there exists a price sequence* $\langle p(t) \rangle$ *in* $R^n_+$ *such that*

(i) $\langle x(t), y(t), p(t) \rangle$ *is competitive,*

(ii) $\delta^t V(y(t)) - p(t)\, y(t) \geqslant \delta^t V(y) - p(t)\, y \qquad$ *for* $y$ *in* $R^n_+$, $t \geqslant 0$, $\quad$ (2.7)

*and*

(iii) $\lim_{t \to \infty} p(t)\, x(t) = 0.$ $\quad$ (2.8)

Conditions (2.7) and (2.8) in the above result are not "independent." For a competitive program, (2.7) implies (2.8), and (2.8) implies (2.7). We note this formally in the next result.

PROPOSITION 2.3. *Under* (A.1)–(A.4) *and* (A.6), *if* $\langle x(t), y(t), p(t) \rangle$ *is a competitive program from* $\tilde{y}$ *in* $R^n_+$, *then it satisfies* (2.7) *if and only if it satisfies* (2.8).

*Proof.* If $\langle x(t), y(t), p(t) \rangle$ is competitive and satisfies (2.7), then $\delta^t V(y(t)) - p(t)\, y(t) \geqslant \delta^t V(0) - p(t)\, 0$. Hence $0 \leqslant p(t)\, y(t) \leqslant \delta^t [V(y(t)) - V(0)]$. Now, $V(y(t))$ is bounded above (by Lemma 2.2 and (A.6)), and $V$ is defined over $R^n_+$ (by Lemma 2.3). Consequently, we have $\lim_{t \to \infty} p(t)\, y(t) = 0$, which implies (2.8), since $0 \leqslant x(t) \leqslant y(t)$ and $p(t) \geqslant 0$.

If $\langle x(t), y(t), p(t) \rangle$ is competitive and satisfies (2.8), then we verify (2.7) as follows. Pick any $T \geqslant 0$ and $y$ in $R^n_+$. Let $\langle x'(s), y'(s) \rangle$ be any program from $y$. Since $\langle x(t), y(t), p(t) \rangle$ is competitive, so by (2.4), we have for $t \geqslant T$,

$$\delta^t [w(c'(t-T)) - w(c(t))] \leqslant [p(t+1)\, y(t+1) - p(t)\, y(t)] - [p(t+1)\, y'(t-T+1) - p(t)\, y'(t-T)].$$

Summing this from $t = T$ to $t = T + N$, we have

$$\begin{aligned} \sum_{T}^{T+N} \delta^t [w(c'(t-T)) - w(c(t))] &\leqslant [p(T+N+1)\, y(T+N+1) - p(T)\, y(T)] \\ &\quad - [p(T+N+1)\, y'(N+1) - p(T)\, y'(0)] \\ &\leqslant p(T+N+1)\, y(T+N+1) - p(T)\, y(T) + p(T)\, y'(0). \end{aligned}$$

Thus, we have

$$\delta^T \left[ \sum_{0}^{N} \delta^s w(c'(s)) - \sum_{T}^{T+N} \delta^{t-T} w(c(t)) \right] \leqslant p(T+N+1)\, y(T+N+1) - p(T)\, y(T) + p(T)\, y.$$

Since the sums in the above expressions converge as $N \to \infty$ [by (A.1)–(A.4) and (A.6)], and (2.8) holds, so

$$\delta^T \left[ \sum_0^\infty \delta^s w(c'(s)) - \sum_T^\infty \delta^{t-T} w(c(t)) \right] \leqslant p(T)\, y - p(T)\, y(T).$$

By the principle of optimality, $\langle x(T+s), y(T+s) \rangle$ is an optimal program from $y(T)$, and so

$$\sum_T^\infty \delta^{t-T} w(c(t)) = \sum_0^\infty \delta^s w(c(T+s)) = V(y(T)).$$

Consequently we have

$$\delta^T V(y) - \delta^T V(y(T)) \leqslant p(T)\, y - p(T)\, y(T),$$

which establishes the inequality in (2.7) for the given $y$ and $T$. Since $y$ in $R^n_+$ and $T \geqslant 0$ were arbitrary, so (2.7) is satisfied. ∎

2f. *An Optimal Stationary Program*

A program $\langle x^*(t), y^*(t) \rangle$ from $y^*$ is a *stationary program* if there is $x^*$ such that $(x^*(t), y^*(t)) = (x^*, y^*)$ for $t \geqslant 0$. It is an *optimal stationary program* (abbreviated as o.s.p.) if it is a stationary program and it is optimal from $y^*$. In this case, we refer to $x^*$ as an *optimal stationary stock* (abbreviated as o.s.s.). We refer to an o.s.p. as $\langle x^*, y^* \rangle$ with obvious interpretation, and its associated consumption sequence as $\langle c^* \rangle$, where $c^* \equiv y^* - x^*$. A vector $q^*$ in $R^n_+$ is a *stationary price support* for an o.s.p. $\langle x^*, y^* \rangle$ if $\langle x^*(t), y^*(t), p^*(t) \rangle \equiv \langle x^*, y^*, \delta^t q^* \rangle$ is competitive from $y^*$.

The question of existence of an o.s.p. has been discussed extensively in the literature, most recently by Khan and Mitra [6] and McKenzie [7]. We note below a proposition on the existence of an optimal stationary program *with stationary price support.* A version of this result has been established by Peleg and Ryder [11], under somewhat different assumptions than the ones we use. The version we report here can be obtained from the result of Khan and Mitra [6], and is proved in Dasgupta and Mitra [4].

The technology set $\Omega$ is called *δ-productive* if there exists $(\hat{x}, \hat{y})$ in $\Omega$, such that $\delta \hat{y} \gg \hat{x}$. We shall need

(A.10) *$\Omega$ is δ-productive.*

PROPOSITION 2.4. *Under* (A.1)–(A.8) *and* (A.10), *there is* $(x^*, y^*)$ *in* $\Omega$, $c^* \equiv y^* - x^* > 0$, $w(c^*) > w(0)$, *and* $q^*$ *in* $R^n_+$ *such that* $\langle x^*, y^* \rangle$ *is an o.s.p. with stationary price support,* $q^*$.

For the remainder of the paper we fix one particular o.s.p. $\langle x^*, y^* \rangle$ and its stationary price support $q^*$ for the purpose of definitions to follow. Furthermore, we denote $\delta^t q^*$ by $p^*(t)$ for $t \geqslant 0$.

## 3. Characterization of Optimality of Competitive Programs in Terms of a Decentralizable Condition

In this section, we show that optimality of competitive programs can be characterized in terms of the simple decentralizable rule of Brock and Majumdar [1], which requires for its verification in each period knowledge of current prices (that is, measured in terms of current welfare) and output quantities and knowledge of $(x^*, y^*, q^*)$. Clearly, the difficult part of this characterization is to show that if a competitive program satisfies the decentralizable rule [that the scalar product of the difference of prices and quantities, between those of the given competitive program and those of the o.s.p., be non-positive, period by period] then it is optimal. We first establish two preliminary results (Lemmas 3.1 and 3.2) and then provide the main theorem of the paper (Theorem 3.1).

If $\langle x(t), y(t), p(t) \rangle$ is a competitive program from $\tilde{y}$ in $R^n_+$, then we denote

$$\mu(t) \equiv (p(t) - p^*(t))\,(x(t) - x^*(t)) \qquad \text{for} \quad t \geqslant 0$$

$$v(t) \equiv (p(t) - p^*(t))\,(y(t) - y^*(t)) \qquad \text{for} \quad t \geqslant 0$$

$$\theta(t) \equiv \mu(t+1) - \mu(t) \qquad \text{for} \quad t \geqslant 0.$$

Furthermore, we denote the *current price sequence* associated with it, $\langle p(t)/\delta^t \rangle$, by $\langle q(t) \rangle$.

Lemma 3.1. *Suppose $\langle x(t), y(t), p(t) \rangle$ is a competitive program from $\tilde{y}$. Then the following conditions hold*:

(i) $v(t+1) \geqslant \mu(t) \geqslant v(t)$ *for* $t \geqslant 0$,

(ii) $\mu(t+1) \geqslant \mu(t)$ *for* $t \geqslant 0$.

*Proof.* Since $\langle x(t), y(t), p(t) \rangle$ is competitive, so by (2.2), we have for $t \geqslant 0$,

$$\delta^t[w(c(t)) - w(c^*)] \geqslant p(t)[c(t) - c^*]. \tag{3.1}$$

Since $\langle x^*, y^*, p^*(t) \rangle$ is competitive, so by (2.2), we have for $t \geqslant 0$,

$$\delta^t[w(c^*) - w(c(t))] \geqslant -p^*(t)[c(t) - c^*]. \tag{3.2}$$

Adding (3.1) and (3.2), we get

$$[p(t)-p^*(t)][c(t)-c^*]\leqslant 0 \qquad \text{for} \quad t\geqslant 0,$$

so that

$$[p(t)-p^*(t)][x(t)-x^*]\geqslant [p(t)-p^*(t)][y(t)-y^*] \qquad \text{for} \quad t\geqslant 0. \quad (3.3)$$

Similarly, since $\langle x(t), y(t), p(t)\rangle$ is competitive, so by (2.3), we have for $t\geqslant 0$,

$$p(t+1)[y(t+1)-y^*]\geqslant p(t)[x(t)-x^*]. \qquad (3.4)$$

And, since $\langle x^*, y^*, p^*(t)\rangle$ is competitive, so by (2.3), we have for $t\geqslant 0$,

$$-p^*(t+1)[y(t+1)-y^*]\geqslant -p^*(t)[x(t)-x^*]. \qquad (3.5)$$

Adding (3.4) and (3.5), we get for $t\geqslant 0$,

$$[p(t+1)-p^*(t+1)][y(t+1)-y^*]\geqslant [p(t)-p^*(t)][x(t)-x^*]. \qquad (3.6)$$

Combining (3.3) and (3.6) yields (i).

To establish (ii), note simply from (i) that $\mu(t+1)\geqslant v(t+1)$ for $t\geqslant 0$, and $v(t+1)\geqslant \mu(t)$ for $t\geqslant 0$, so that the conclusion is obvious. ∎

LEMMA 3.2. *Suppose $\langle x(t), y(t), p(t)\rangle$ is a competitive program from $\tilde{y}$, which satisfies $\mu(t)\leqslant 0$ for $t\geqslant 0$. Then, the following conditions hold:*

(i) $\theta(t)\geqslant 0$ *for* $t\geqslant 0$, *and* $\sum_0^\infty \theta(t)<\infty$,

(ii) $\theta(t)\to 0$ *as* $t\to\infty$.

*Proof.* From Lemma 3.1(ii), we have $\theta(t)\geqslant 0$ for $t\geqslant 0$. Also, for $T\geqslant 0$,

$$S(T)=\sum_0^T \theta(t)=\sum_0^T [\mu(t+1)-\mu(t)]=\mu(T+1)-\mu(0)\leqslant -\mu(0)$$

since $\mu(T+1)\leqslant 0$, by hypothesis. Thus, $\langle S(T)\rangle$ is a monotonically non-decreasing sequence, bounded above, hence it converges. This establishes (i).

It follows directly from (i) that $\theta(t)\to 0$ *as* $t\to\infty$, which is (ii). ∎

A vector $x$ in $R_+^n$ is called *expansible* if there is $y\gg x$, such that $(x, y)$ is in $\Omega$. It is called *proportionately expansible* if there is $\lambda>1$, such that $(x, \lambda x)$ is in $\Omega$. Clearly, if $x$ is expansible, it is also proportionately expansible; the converse is false.

THEOREM 3.1. *Suppose $x^*$ is proportionately expansible. If $\langle x(t), y(t), p(t)\rangle$ is a competitive program from $\tilde{y}$, which satisfies*

$$(q(t)-q^*)(y(t)-y^*)\leqslant 0 \qquad \textit{for} \quad t\geqslant 0, \tag{3.7}$$

*then $\langle x(t), y(t)\rangle$ is an optimal program from $\tilde{y}$.*

*Proof.* We note right at the outset that (3.7) implies that $v(t)\leqslant 0$ for $t\geqslant 0$. This, in turn, implies that $\mu(t)\leqslant 0$ for $t\geqslant 0$ by Lemma 3.1, and so $\theta(t)\to 0$ as $t\to\infty$ by Lemma 3.2.

Our first task now is to show that

$$p(t)\,x^*\to 0 \qquad \text{as} \quad t\to\infty. \tag{3.8}$$

To this end, use the competitive conditions for $\langle x(t), y(t), p(t)\rangle$ to write for $t\geqslant 0$,

$$\begin{aligned}\delta^{t+1}w(c(t+1))&+p(t+1)\,x(t+1)-p(t)\,x(t)\\&\geqslant \delta^{t+1}w(c)+p(t+1)(y-c)-p(t)\,x\end{aligned} \tag{3.9}$$

for all $(x, y)$ in $\Omega$, and all $c$ in $R^n_+$. Since $x^*$ is proportionately expansible, there is $\lambda>1$ such that $(x^*, \lambda x^*)$ is in $\Omega$. Using this in (3.9), we get for $t\geqslant 0$,

$$\begin{aligned}\delta^{t+1}w(c(t+1))&+p(t+1)\,x(t+1)-p(t)\,x(t)\\&\geqslant \delta^{t+1}w(0)+p(t+1)\,\lambda x^*-p(t)\,x^*.\end{aligned} \tag{3.10}$$

Transposing terms in (3.10), we get for $t\geqslant 0$

$$\begin{aligned}\delta^{t+1}[w(c(t+1))-w(0)]&+p(t+1)[x(t+1)-x^*]-p(t)[x(t)-x^*]\\&\geqslant (\lambda-1)\,p(t+1)\,x^*.\end{aligned} \tag{3.11}$$

Note that $p(t+1)[x(t+1)-x^*]=[p(t+1)-p^*(t+1)][x(t+1)-x^*]+p^*(t+1)[x(t+1)-x^*] = \mu(t+1)+p^*(t+1)[x(t+1)-x^*]$. Similarly, $p(t)[x(t)-x^*]=\mu(t)+p^*(t)[x(t)-x^*]$. Using these in (3.11), we get

$$\begin{aligned}\delta^{t+1}[w(c(t+1))-w(0)]&+\mu(t+1)+p^*(t+1)[x(t+1)-x^*]\\&-\mu(t)-p^*(t)[x(t)-x^*]\geqslant (\lambda-1)\,p(t+1)\,x^*.\end{aligned} \tag{3.12}$$

Looking at the left-hand side of (3.12), we note that $[\mu(t+1)-\mu(t)]\equiv\theta(t)$ converges to zero as $t\to\infty$. The term $p^*(t)[x(t)-x^*]$ clearly converges to zero, since $p^*(t)=\delta^t q^*\to 0$ as $t\to\infty$, and $\|x(t)\|\leqslant B$; the same observation holds for the term $p^*(t+1)\,[x(t+1)-x^*]$. Finally, $\|c(t+1)\|\leqslant B$ and continuity of $w$ [by (A.6)] imply that $w(c(t+1))$ is bounded above, while $w(0)$ is in $R$; so $\delta^{t+1}[w(c(t+1))-w(0)] \to 0$ as $t \to \infty$, since $\delta^{t+1} \to 0$ as $t \to \infty$. Thus, (3.8) follows from (3.12).

Our second task is to show that

$$p(t)\,x(t) \to 0 \qquad \text{as} \quad t \to \infty. \tag{3.13}$$

This is rather easy, given (3.8). To see this, note that since $\mu(t) \leqslant 0$ for $t \geqslant 0$, so

$$p(t)(x(t)-x^*) \leqslant p^*(t)(x(t)-x^*) \qquad \text{for} \quad t \geqslant 0. \tag{3.14}$$

This yields in turn, for $t \geqslant 0$,

$$p(t)\,x(t) \leqslant p(t)\,x^* + p^*(t)(x(t)-x^*) \leqslant p(t)\,x^* + p^*(t)\,x(t). \tag{3.15}$$

Now, $p^*(t)\,x(t) \to 0$ as $t \to \infty$, since $p^*(t) = \delta^t q^* \to 0$ as $t \to \infty$, while $\|x(t)\| \leqslant B$ for $t \geqslant 0$. Also, by (3.8), $p(t)\,x^* \to 0$ as $t \to \infty$. Now, (3.13) follows from (3.15).

Since $\langle x(t), y(t), p(t) \rangle$ is a competitive program satisfying the transversality condition given by (3.13), so $\langle x(t), y(t) \rangle$ is optimal from $\tilde{y}$, by Proposition 2.1. ∎

*Remark.* The proof of Theorem 3.1 combines the ideas used in the earlier proofs (in Theorems 4.1 and 4.3) of Dasgupta and Mitra [3]. However, it uses weaker assumptions than either of those earlier results, and therefore includes both those results as special cases.

If we look at the proof of Theorem 3.1, we notice, in particular, that it does not use the convexity of the technology set or concavity of the welfare function. In fact, if we assume (A.1)–(A.4) and (A.6), and we also assume the condition

(E) *There exists an o.s.p. $(x^*, y^*)$, with stationary price support, $q^*$, such that $x^*$ is proportionately expansible,*

then Theorem 3.1 holds. In this sense, the requirements to prove Theorem 3.1 are similar to the requirements to prove Proposition 2.1. The *additional requirement* is condition (E), which can then be considered to be the "price paid" for replacing the transversality condition in Proposition 2.1 by the decentralizable condition in Theorem 3.1.

The converse of Theorem 3.1 is fairly well known. We state and prove it here formally for the sake of completeness.

THEOREM 3.2. *Suppose $\langle x(t), y(t) \rangle$ is an optimal program from $\tilde{y}$ in*

$R^n_{++}$. *Then, there is a sequence* $\langle p(t)\rangle$ *such that* $\langle x(t), y(t), p(t)\rangle$ *is competitive from* $\tilde{y}$, *and*

$$(q(t)-q^*)(y(t)-y^*)\leqslant 0 \qquad \text{for} \quad t\geqslant 0. \tag{3.16}$$

*Proof.* By Proposition 2.2, there is a sequence $\langle p(t)\rangle$ such that $\langle x(t), y(t), p(t)\rangle$ is competitive from $\tilde{y}$, and for $t\geqslant 0$,

$$V(y(t))-q(t)\,y(t)\geqslant V(y)-q(t)\,y \qquad \text{for all } y \text{ in } R^n_+. \tag{3.17}$$

Using $y=y^*$ in (3.17), we get for $t\geqslant 0$

$$V(y(t))-V(y^*)\geqslant q(t)[y(t)-y^*]. \tag{3.18}$$

Using Proposition 2.4, $\langle x^*, y^*, p^*(t)\rangle$ is competitive from $y^*$, and $p^*(t)\,x^*=\delta^t q^* x^*\to 0$ as $t\to\infty$. Hence, by Proposition 2.3,

$$V(y^*)-q^*y^*\geqslant V(y)-q^*y \qquad \text{for all } y \text{ in } R^n_+. \tag{3.19}$$

Using $y=y(t)$ in (3.19), and transposing terms,

$$V(y^*)-V(y(t))\geqslant -q^*[y(t)-y^*]. \tag{3.20}$$

Adding (3.18) and (3.20), we get (3.16). ∎

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